

# Interaction solutions for mKP equation with nonlocal symmetry reductions and CTE method

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The nonlocal symmetries for the modified Kadomtsev-Petviashvili (mKP) equation are obtained with the truncated Painlevé method. The nonlocal symmetries can be localized to the Lie point symmetries by introducing auxiliary dependent variables. The finite symmetry transformations and similarity reductions related with the nonlocal symmetries are computed. The multi-solitary wave solution and interaction solutions among a soliton and cnoidal waves of the mKP equation are presented. In the meanwhile, the consistent tanh expansion (CTE) method is applied to the mKP equation. The explicit interaction solutions among a soliton and other types of nonlinear waves such as cnoidal periodic waves and multiple resonant soliton solutions are given.

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## I. INTRODUCTION

The study of nonlinear integrable systems is one of the most important subjects in modern physics and nonlinear science. The explicit solutions of the nonlinear integrable equations are very important due to their wide applications in explaining physical phenomena. The most effective methods are the inverse scattering transformation [1], bilinear form [2], symmetry reduction [3], Darboux transformation [4], Painlevé analysis method [5], Bäcklund transformation (BT) [6], separated variable method [7], etc. For these methods, the interaction solutions between solitons and nonlinear waves are difficult to obtain. Recently, Lou and his colleagues proposed the localization procedure related with the nonlocal symmetry [8, 9] and the consistent tanh expansion (CTE) method [9, 10] to find interaction solutions. Some interesting interaction solutions between a soliton and the cnoidal waves, Painlevé waves, Airy waves, Bessel waves are generated with the methods [8–16].

In this letter, we focus on the modified Kadomtsev-Petviashvili (mKP) equation [17]

$$u_t - \frac{1}{4}u_{xxx} - \frac{3}{4}w_y + \frac{3}{2}u^2u_x + \frac{3}{2}u_xw = 0, \quad (1a)$$

$$u_y = w_x, \quad (1b)$$

which describes water waves in  $(x, y)$  plane when the nonlinearity is higher than for the KP equation. The integrable properties of this equation such as the Lax pair [17, 18], Darboux transformation [19, 20] and explicit solutions [21, 22] have been obtained. From the mKP equation, a new integrable system is given by means of an asymptotically exact reduction method [23].

The paper is organized as follows. In section 2, the nonlocal symmetries for the mKP equation are obtained with the truncated Painlevé method. To solve the initial value problem of the nonlocal

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symmetries, the nonlocal symmetries are localized by prolongation the mKP equation. The finite symmetry transformations are obtained by solving the initial value problem of the Lie's first principle. The multi-solitary wave solution of the mKP equation is obtained using the finite symmetry transformations. In section 3, the group invariant solutions related to the nonlocal symmetries are studied with the symmetry reduction method. The corresponding explicit interaction solutions among a soliton and cnoidal waves are given. Section 4 is devoted to the CTE approach for the mKP equation. Some exact interaction solutions among different nonlinear excitations such as cnoidal periodic waves and multiple resonant soliton solutions are explicitly given. The last section is a simple summary and discussion.

## II. NONLOCAL SYMMETRIES OF THE MKP EQUATION AND EXPLICIT SOLUTIONS

To construct the BT of the mKP equation, we truncate the Laurent series as [5]

$$u = \frac{u_0}{\phi} + u_1, \quad w = \frac{w_0}{\phi} + w_1, \quad (2)$$

where the function  $\phi(x, y, t) = 0$  is the equation of singularity manifold, the functions  $u_0$ ,  $u_1$ ,  $w_0$  and  $w_1$  are determined by substituting of expansion (2) into (1) and balancing all coefficients of each power of  $\phi$  independently. Then, we get

$$u_0 = \phi_x, \quad w_0 = \phi_y, \quad (3)$$

$$u_1 = -\frac{1}{2} \left( \frac{\phi_{xx}}{\phi_x} + \frac{\phi_y}{\phi_x} \right), \quad w_1 = -\frac{2}{3} \frac{\phi_t}{\phi_x} - \frac{\phi_{xy}}{2\phi_x} + \frac{\phi_{xxx}}{6\phi_x} + \frac{1}{4} \left( \frac{\phi_y^2}{\phi_x^2} - \frac{\phi_{xx}^2}{\phi_x^3} \right). \quad (4)$$

Substituting the expressions (2), (3) and (4) into (1), the field  $\phi$  satisfy the following Schwarzian mKP form

$$4 \left( \frac{\phi_t}{\phi_x} \right)_x - \frac{\partial}{\partial x} \{ \phi; x \} - 3 \left( \frac{\phi_y}{\phi_x} \right)_y - \frac{3}{2} \left( \frac{\phi_y^2}{\phi_x^2} \right)_x = 0, \quad (5)$$

where  $\{ \phi; x \} = \frac{\partial}{\partial x} \left( \frac{\phi_{xx}}{\phi_x} \right) - \frac{1}{2} \left( \frac{\phi_{xx}}{\phi_x} \right)^2$  is the Schwarzian derivative. For the BT (2), two pairs of functions  $u, w$  and  $u_1, w_1$  satisfy (1). The latter solutions  $u_1, w_1$  are related to  $\phi$  with (4).

The nonlocal symmetries of the mKP equation (1) can read out from the BT [9]

$$\sigma^u = \phi_x, \quad \sigma^w = \phi_y. \quad (6)$$

The nonlocal symmetries (6) are the residual of the singularity manifold  $\phi$ . Thus, this nonlocal symmetries are called as the residual symmetries (RS) [9]. Besides, this nonlocal symmetries can also be obtained from the Schwarzian form (5) [24]. The Schwarzian form (5) is invariant under the Möbius transformation

$$\phi \rightarrow \frac{a\phi + b}{c\phi + d}, \quad ac \neq bd, \quad (7)$$

which means (5) possesses the symmetry  $\sigma^\phi = -\phi^2$  in special case  $a = d = 1$ ,  $b = 0$ ,  $c = \epsilon$ . The nonlocal symmetries (6) will be obtained with substituting the Möbius transformation symmetry  $\sigma^\phi$  into the linearized equation of (4).

For the nonlocal symmetries (6), the corresponding initial value problem is

$$\begin{aligned}\frac{d\bar{u}}{d\epsilon} &= \phi_x, & \bar{u}|_{\epsilon=0} &= u, \\ \frac{d\bar{w}}{d\epsilon} &= \phi_y, & \bar{w}|_{\epsilon=0} &= w.\end{aligned}\tag{8}$$

It is difficult to solve the initial value problem of the Lie's first principle (8) due to the intrusion of the function  $\phi$  and its differentiations [9]. To solve the initial value problem (8), we prolong the mKP system (1) such that RS become the local Lie point symmetries for the prolonged system. By localization the nonlocal symmetries (6), the potential fields of  $\phi$  are introduced as

$$\phi_x = g, \tag{9}$$

$$\phi_y = h. \tag{10}$$

It is easy to verify that the local Lie point symmetries for the prolonged systems (1), (4), (9) and (10) read as

$$\sigma^u = g, \quad \sigma^w = h, \quad \sigma^\phi = -\phi^2, \quad \sigma^g = -2\phi g, \quad \sigma^h = -2\phi h. \tag{11}$$

Correspondingly, the initial value problem becomes

$$\begin{aligned}\frac{d\bar{u}}{d\epsilon} &= g, & \bar{u}|_{\epsilon=0} &= u, \\ \frac{d\bar{w}}{d\epsilon} &= h, & \bar{w}|_{\epsilon=0} &= w, \\ \frac{d\bar{\phi}}{d\epsilon} &= -\phi^2, & \bar{\phi}|_{\epsilon=0} &= \phi, \\ \frac{d\bar{g}}{d\epsilon} &= -2\phi g, & \bar{g}|_{\epsilon=0} &= g, \\ \frac{d\bar{h}}{d\epsilon} &= -2\phi h, & \bar{h}|_{\epsilon=0} &= h.\end{aligned}\tag{12}$$

The solution of the initial value problem (12) for the enlarged mKP system (1), (4), (9) and (10) can be written as

$$\bar{u} = u + \frac{\epsilon g}{\epsilon\phi + 1}, \quad \bar{w} = w + \frac{\epsilon h}{\epsilon\phi + 1}, \quad \bar{\phi} = \frac{\phi}{\epsilon\phi + 1}, \quad \bar{g} = \frac{g}{(\epsilon\phi + 1)^2}, \quad \bar{h} = \frac{h}{(\epsilon\phi + 1)^2}. \tag{13}$$

Using the finite symmetry transformations (13), one can obtain a new solution from any initial solution. For example, we take the trivial solution  $u = w = 0$  for (1). The multi-solitary wave solution for (4) is supposed as [25]

$$\phi = 1 + \sum_{n=1}^N \exp(k_n x + l_n y + \omega_n t), \tag{14}$$

where  $k_n, l_n$  and  $\omega_n$  are arbitrary constants. The multi-solitary wave solution (14) is the solution of (4) and (5) only with the relations

$$l_n = -k_n^2, \quad \omega_n = k_n^3. \tag{15}$$

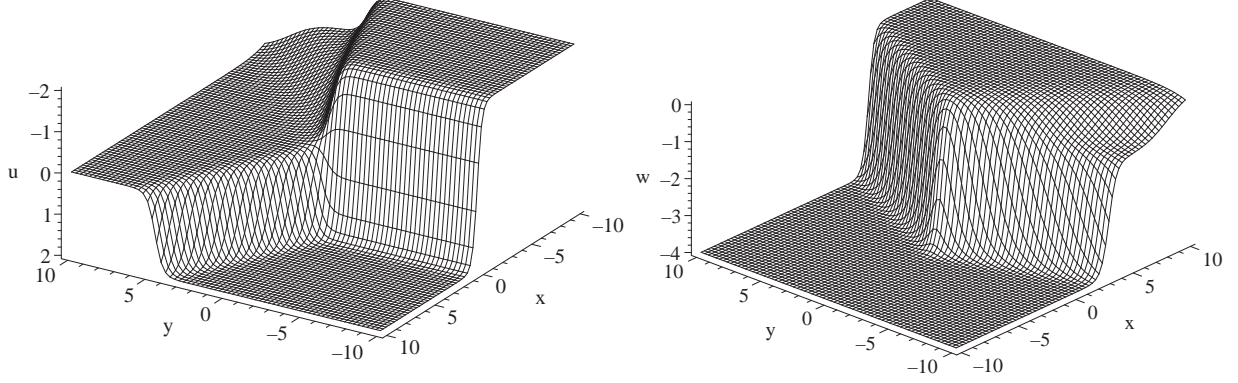


FIG. 1: Three-solitary wave solution for the fields  $u$  and  $w$  with the parameters  $n = 3$ ,  $\epsilon = -\frac{1}{4}$ ,  $k_1 = -1$ ,  $k_2 = -2$ ,  $k_3 = 3$  and  $t = 0$  respectively.

A solution of equation (1) presents in the following form using (9), (10) and (13)

$$u = \frac{\sum_{n=1}^N \epsilon k_n \exp(k_n x + l_n y + \omega_n t)}{-1 + \epsilon + \sum_{n=1}^N \epsilon \exp(k_n x + l_n y + \omega_n t)}, \quad (16a)$$

$$w = -\frac{\sum_{n=1}^N \epsilon k_n^2 \exp(k_n x + l_n y + \omega_n t)}{-1 + \epsilon + \sum_{n=1}^N \epsilon \exp(k_n x + l_n y + \omega_n t)}. \quad (16b)$$

Figure 1 shows the three-solitary wave solution of the fields  $u$  and  $w$  with the parameters  $n = 3$ ,  $\epsilon = -\frac{1}{4}$ ,  $k_1 = -1$ ,  $k_2 = -2$ ,  $k_3 = 3$ .

### III. SIMILARITY REDUCTIONS WITH THE NONLOCAL SYMMETRIES

In this section, the symmetry reductions related to the nonlocal symmetries will be discussed. According to the standard Lie symmetry approach, the Lie point symmetries for the prolonged systems possess the form

$$\begin{aligned} \sigma^u &= Xu_x + Yu_y + Tu_t - U, \\ \sigma^w &= Xw_x + Yw_y + Tw_t - W, \\ \sigma^\phi &= X\phi_x + Y\phi_y + T\phi_t - \Phi, \\ \sigma^g &= Xg_x + Yg_y + Tg_t - G, \\ \sigma^h &= Xh_x + Yh_y + Th_t - H, \end{aligned} \quad (17)$$

where  $X, Y, T, U, W, \Phi, G, H$  are functions of  $x, t, u, w, \phi, g, h$ .

The symmetries  $\sigma^k$  ( $k = u, w, \phi, g, h$ ) are defined as the solution of the linearized equations of the prolonged systems (1) (4), (9) and (10)

$$\sigma_t^u - \frac{1}{4}\sigma_{xxx}^u - \frac{3}{4}\sigma_y^w + 3\sigma^u u u_x + \frac{3}{2}u^2 \sigma_x^u + \frac{3}{2}\sigma_x^u w + \frac{3}{2}\sigma^w u_x = 0, \quad (18a)$$

$$\sigma_y^u - \sigma_x^w = 0, \quad (18b)$$

$$\sigma^u + \frac{1}{2}\frac{\sigma_y^u}{\phi_x} + \frac{1}{2}\frac{\sigma_{xx}^u}{\phi_x} - \frac{1}{2}\frac{\sigma_x^u \phi_y}{\phi_x^2} - \frac{1}{2}\frac{\sigma_x^u \phi_{xx}}{\phi_x^2} = 0, \quad (18c)$$

$$\sigma^w + \frac{4\sigma_t^w + 3\sigma_{xy}^w - \sigma_{xxx}^w}{6\phi_x} - \frac{2}{3}\frac{\sigma_x^w \phi_t}{\phi_x^2} - \frac{\sigma_y^w \phi_y}{2\phi_x^2} - \frac{\sigma_x^w \phi_{xy}}{2\phi_x^2} + \frac{\sigma_x^w \phi_{xxx}}{6\phi_x^2} + \frac{\sigma_{xx}^w \phi_{xx}}{2\phi_x^2} + \frac{\sigma_x^w \phi_y^2 - \sigma_x^w \phi_{xx}^2}{2\phi_x^3} = 0, \quad (18d)$$

$$\sigma_x^\phi - \sigma^g = 0, \quad (18e)$$

$$\sigma_y^\phi - \sigma^h = 0. \quad (18f)$$

That is to say, the prolong systems are invariant under transformations

$$u \rightarrow u + \epsilon \sigma^u, \quad w \rightarrow w + \epsilon \sigma^w, \quad \phi \rightarrow \phi + \epsilon \sigma^\phi, \quad g \rightarrow g + \epsilon \sigma^g, \quad h \rightarrow h + \epsilon \sigma^h, \quad (19)$$

with the infinitesimal parameter  $\epsilon$ .

Substituting (17) into the symmetry equations (18) and requiring  $u, w, \phi, g, h$  to satisfy the prolonged systems, the determining equations are obtained with collecting the coefficients of  $u, w, \phi, g, h$  and its derivatives. The infinitesimals  $X, Y, T, U, W, \Phi, G$  and  $H$  are given by solving the determining equations

$$\begin{aligned} X &= \frac{1}{3}f_{1t}x + \frac{2}{3}f_{2t}y + \frac{2}{9}f_{1tt}y^2 + f_3, \quad Y = \frac{2}{3}f_{1t}y + f_2, \quad T = f_1t + C_1, \quad G = 2C_2\phi g + C_3g - \frac{1}{3}f_{1t}g, \\ U &= -\frac{1}{3}f_{1t}u - C_2g + \frac{2}{9}f_{1tt}y + \frac{1}{3}f_{2t}, \quad H = 2C_2\phi h + C_3h - \frac{2}{3}f_{1t}h - \frac{2}{3}f_{2t}g - \frac{4}{9}f_{1tt}gy, \\ W &= -\frac{2}{3}f_{1t}w - \frac{2}{3}f_{2t}u - \frac{4}{9}f_{1tt}yu - C_2h + \frac{2}{9}f_{1tt}x + \frac{4}{27}f_{1ttt}y^2 + \frac{4}{9}f_{2tt}y + \frac{2}{3}f_{3t}, \quad \Phi = C_2\phi^2 + C_3\phi + C_4, \end{aligned} \quad (20)$$

where  $f_1, f_2$  and  $f_3$  are arbitrary functions of  $t$  and  $C_1, C_2, C_3$  and  $C_4$  are arbitrary constants. It is well known that whence a symmetry is known, the related group invariant solutions can be naturally obtained with the symmetry constraint condition  $\sigma^k = 0$  defined by (17). It is equivalent to solving the following characteristic equations

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dt}{T} = \frac{du}{U} = \frac{dw}{W} = \frac{d\phi}{\Phi} = \frac{dg}{G} = \frac{dh}{H}. \quad (21)$$

To solve the characteristic equations, two special cases are listed in the following.

**Case I.**  $f_1 = 0$ . In this case, without loss of generality, we rewrite the arbitrary functions  $f_2$  and  $f_3$  as

$$f_2 = C_1 m_{1t}, \quad f_3 = C_1 m_{2t}, \quad (22)$$

where  $m_1$  and  $m_2$  being arbitrary functions of  $t$ . We can find the similarity solutions after solving out the characteristic equations (21)

$$\phi = \frac{\Delta}{2C_2}M - \frac{C_3}{2C_2}, \quad \Delta = \sqrt{4C_2C_4 - C_3^2}, \quad (23a)$$

$$g = G(\sec M)^2, \quad (23b)$$

$$h = (H - \frac{2}{3}m_{1t}G)(\sec M)^2, \quad (23c)$$

$$u = U + \frac{2C_2}{\Delta}G(\arctan M - M) - \frac{C_2}{C_1}tG + \frac{1}{3}m_{1t}, \quad (23d)$$

$$\begin{aligned} w &= W + \frac{2C_2}{\Delta}H(\arctan M - M) - \frac{4C_2}{3\Delta}m_{1t}G(\arctan M - M) - \frac{C_2}{C_1}tH \\ &\quad - \frac{2}{3}m_{1t}U + \frac{2C_2}{3C_1}tm_{1t}G + \frac{4}{9}m_{1tt}y - \frac{1}{9}m_{1t}^2 + \frac{2}{3}m_{2t}, \end{aligned} \quad (23e)$$

with the similarity variables  $\xi = x - \frac{2}{3}m_{1t}y - m_2$  and  $\eta = y - m_1$  and  $M = \tan(\frac{\Delta}{2C_1}(\Phi + t))$  for similarity. Substituting (23) into (9), (10), (4) and (5), the invariant functions  $G, H, U, W$  and  $\Phi$  satisfy the reduction systems

$$G = \frac{\Delta^2}{4C_1C_2}\Phi_\xi, \quad (24a)$$

$$H = \frac{\Delta^2}{4C_1C_2}\Phi_\eta, \quad (24b)$$

$$U = \frac{\Delta^2}{4C_1^2}t\Phi_\xi - \frac{\Delta}{2C_1}\Phi_\xi \arctan M - \frac{\Phi_\eta + \Phi_{\xi\xi}}{2\Phi_\xi}, \quad (24c)$$

$$W = \frac{2}{9}m_{1t}^2 + \frac{\Delta^2}{12C_1^2}\Phi_\xi^2 + \frac{\Delta^2}{4C_1^2}t\Phi_\eta + \frac{\Phi_{\xi\xi\xi} - 3\Phi_{\xi\eta} - 4}{6\Phi_\xi} + \frac{\Phi_\eta^2 - \Phi_{\xi\xi}^2}{4\Phi_\xi^2} - \frac{\Delta}{2C_1}\Phi_\eta \arctan M, \quad (24d)$$

$$\frac{\Delta^2}{C_1^2}\Phi_{\xi\xi}\Phi_\xi^4 + \Phi_\xi^2\Phi_{\xi\xi\xi\xi} + 3\Phi_{\eta\eta}\Phi_\xi^2 - 4\Phi_\xi\Phi_{\xi\xi}\Phi_{\xi\xi\xi} + 3\Phi_{\xi\xi}^3 - 3\Phi_{\xi\xi}\Phi_\eta^2 + 4\Phi_\xi\Phi_{\xi\xi} = 0. \quad (24e)$$

It is obvious that once the solutions  $\Phi$  are solved out with (24e), the fields for  $G$ ,  $H$ ,  $U$  and  $W$  can be solved out directly from (24a)-(24d). The explicit solutions for mKP (1) are immediately obtained by substituting  $\Phi$ ,  $G$ ,  $H$ ,  $U$  and  $W$  and into (23).

For instance, it has a trivial solution  $\Phi = \xi + \eta$  for the system (24e). The exact solution for mKP (1) express as

$$u = -\frac{\Delta}{2C_1}M + \frac{1}{3}m_{1t} - \frac{1}{2}, \quad (25)$$

$$w = \left(-\frac{1}{3}m_{1t} - \frac{1}{2}\right)\frac{\Delta}{C_1}M + \frac{4}{9}m_{1t}(m_1 + \eta) + 9m_{1t}^2 + \frac{1}{3}m_{1t} + \frac{2}{3}m_{2t} + \frac{\Delta^2}{12C_1^2} - \frac{5}{12},$$

which is a nontrivial solution of the mKP equation. Besides, for the reduction equation (24e), its cnoidal wave solution is given

$$\Phi_{1X}^2 - \frac{\Delta^2}{C_1^2}\Phi_1^4 + 2B\Phi_1^3 - 2A\Phi_1^2 + 4\Phi_1 = 0, \quad \Phi_1 = \Phi_X, \quad \Phi(\xi, \eta) = \Phi(\xi + a\eta) = \Phi(X), \quad (26)$$

where arbitrary constants  $a, A, B$ . The solution for (26) can be explicitly expressed by Jacobi elliptic functions

$$\Phi_1 = -\frac{r_1r_3S^2}{r_1S^2 - r_1 + r_3}, \quad S = \operatorname{sn}\left(\frac{\Delta}{2C_1}\sqrt{r_2(r_1 - r_3)}X, m\right), \quad m = \frac{r_1(r_2 - r_3)}{r_2(r_1 - r_3)}, \quad (27)$$

where  $S$  is the Jacobi elliptic function with the modulus  $m$  and the arbitrary constants  $C_1, C_2$  and  $\frac{\Delta}{C_1}$  have been re-expressed by

$$A = \frac{2(r_1r_2 + r_1r_3 + r_2r_3)}{r_1r_2r_3}, \quad B = \frac{2(r_1 + r_2 + r_3)}{r_1r_2r_3}, \quad \frac{\Delta}{C_1} = \frac{2}{\sqrt{r_1r_2r_3}}.$$

Correspondingly, the field  $\Phi$  has the form

$$\Phi = \frac{2r_3C_1}{\Delta\sqrt{r_2(r_1 - r_3)}}\left(E_\pi\left(S, \frac{r_1}{r_1 - r_3}, m\right) - E_F(S, m)\right), \quad (28)$$

where  $E_F$  and  $E_\pi$  are the first and third incomplete elliptic integrals. It is clear that the exact solution for the mKP equation denotes the interaction between a soliton and cnoidal periodic waves.

**Case II.**  $f_1 = f_2 = C_1 = 0$ . We find the similarity solutions after solving out the characteristic equations (21)

$$\phi = \frac{\Delta}{2C_2}\tan\left(\frac{\Delta}{2f_3}(\phi' + x)\right) - \frac{C_3}{2C_2}, \quad (29a)$$

$$g = g' \sec\left(\frac{\Delta}{2f_3}(\phi' + x)\right)^2, \quad (29b)$$

$$h = h' \sec\left(\frac{\Delta}{2f_3}(\phi' + x)\right)^2, \quad (29c)$$

$$u = u' - \frac{2C_2}{\Delta} g' \tan\left(\frac{\Delta}{2f_3}(\phi' + x)\right), \quad (29d)$$

$$w = w' + \frac{2}{3} \frac{f_3'}{f_3}(\phi' + x) - \frac{2C_2}{\Delta} h' \tan\left(\frac{\Delta}{2f_3}(\phi' + x)\right), \quad (29e)$$

where the group invariant functions  $\phi' = \phi'(y, t)$ ,  $g' = g'(y, t)$ ,  $h' = h'(y, t)$ ,  $u' = u'(y, t)$  and  $w' = w'(y, t)$ . Substituting (29) into (4), (9), (10), the invariant functions  $\phi', g', h', u'$  and  $w'$  satisfy the reduction systems

$$\phi' = -\frac{2f_{3t}}{3f_3}y^2 - 2f_4y + f_5 \quad (30a)$$

$$g' = \frac{\Delta^2}{4C_2f_3}, \quad (30b)$$

$$h' = \frac{\Delta^2\phi'_y}{4C_2f_3}, \quad (30c)$$

$$u' = \frac{2f_{3t}}{3f_3}y + f_4, \quad (30d)$$

$$w' = \frac{4f_{3tt}}{9f_3}y^2 + \frac{4(f_3f_4)_t}{3f_3}y + f_4^2 - \frac{2}{3}f_{5t} + \frac{\Delta^2}{12f_3^2}, \quad (30e)$$

where  $f_4$  and  $f_5$  are arbitrary functions of  $t$ . The explicit solution for the mKP is given with substituting (30) into (29)

$$\begin{aligned} u &= \frac{\Delta}{2f_3} \tan\left(\frac{\Delta}{6f_3^2}(2f_{3t}y^2 + 6f_3f_4y - 3f_3x - 3f_3f_4)\right) + \frac{2f_{3t}y}{3f_3} + f_4, \\ w &= -\Delta\left(\frac{2}{3}\frac{f_{3t}}{f_3^2}y + \frac{f_4}{f_3}\right) \tan\left(\frac{\Delta}{6f_3^2}(2f_{3t}y^2 + 6f_3f_4y - 3f_3x - 3f_3f_4)\right) + \frac{4f_{3t}^2 - 4f_{3tt}f_3}{9f_3^2}y^2 \\ &\quad - \frac{4}{3}f_{4t}y - \frac{2}{3}\frac{f_{3t}}{f_3}x - \frac{4f_{3t}f_5}{9f_3} + \frac{2}{3}f_{5t} - \frac{\Delta^2}{12f_3^2} + f_4^2. \end{aligned} \quad (31)$$

#### IV. CTE METHOD FOR MKP SYSTEM

Recently, the consistent Riccati expansion (CRE)/consistent tanh expansion (CTE) methods are developed to find interaction solutions between solitons and other types of nonlinear waves such as cnoidal waves, Painlevé waves and Airy waves [9, 10, 13–15]. According to the CTE method, we assume the solution for the mKP equation (1) has the generalized truncated tanh expansion form [9, 10]

$$u = u_0 + u_1 \tanh(f), \quad w = w_0 + w_1 \tanh(f), \quad (32)$$

where  $u_0, u_1, w_0, w_1$  and  $f$  are functions of  $(x, y, t)$  and should be determined later. By substituting (32) into the mKP system (1) and vanishing coefficients of all the powers of  $\tanh(f)$ , we can prove the following nonauto-BT theorem after some detail calculations.

**Nonauto-BT theorem.** If  $f$  is the solution of the following equation

$$f_{xxx} + 3f_{yy} - 4f_{xt} - 4f_{xx}f_x^2 + \frac{4f_t f_{xx}}{f_x} - \frac{4f_{xx}f_{xxx}}{f_x} + \frac{3f_{xx}^3}{f_x^2} - \frac{3f_{xx}f_y^2}{f_x^2} = 0, \quad (33)$$

then  $u$  and  $w$  with

$$u = -\frac{f_{xx} + f_y}{2f_x} + f_x \tanh(f), \quad w = -\frac{1}{3}f_x^2 - \frac{2f_t}{3f_x} + \frac{f_{xxx}}{6f_x} - \frac{f_{xy}}{2f_x} - \frac{f_{xx}^2}{4f_x^2} + \frac{f_y^2}{4f_x^2} + f_y \tanh(f), \quad (34)$$

are a solution of the mKP system (1). Once the solutions of (33) are known, the corresponding expression  $u, w$  for (34) can be obtained with the nonauto-BT theorem, whence the new solutions of (1) can be obtained. Here we list three interesting examples.

A quite trivial solution of (33) has the form

$$f = kx + ly + \omega t, \quad (35)$$

where  $k, l$  and  $\omega$  are all the free constants. Substituting the trivial solution (35) into (34), the soliton solution of the mKP system yields

$$u = k \tanh(kx + ly + \omega t) - \frac{l}{2k}, \quad (36)$$

$$w = l \tanh(kx + ly + \omega t) - \frac{k^2}{3} - \frac{2\omega}{3k} + \frac{l^2}{4k^2}. \quad (37)$$

The soliton-cnoidal wave interaction solution for the mKP equation possesses the form

$$f = kx + ly + \omega t + F(k_1x + l_1y + \omega_1t), \quad (38)$$

where  $k_1, l_1$  and  $\omega_1$  are all the free arbitrary constants. Substituting (38) into (34), we have

$$F_{1X}^2 = 4F_1^4 + a_1F_1^3 + a_2F_1^2 + a_3F_1 + a_4, \quad F_1 = F_X, \quad F(k_1x + l_1y + \omega_1t) = F(X), \quad (39)$$

where

$$\begin{aligned} a_1 &= \frac{8k}{k_1} - 2c_2k_1^2, & a_3 &= -\frac{6kl_1^2}{k_1^5} + \frac{4k\omega_1 + 6l_1l}{k_1^4} - \frac{4\omega}{k^3} + 2k(2c_1 - 3c_2k), \\ a_2 &= (2c_1 - 6c_2k)k_1 + \frac{4k^2}{k_1^2}, & a_4 &= -\frac{5k^2l_1^2}{k_1^6} + \frac{4k(k\omega_1 + l_1l)}{k_1^5} + \frac{l^2 - 4k\omega_2}{k_1^4} + \frac{2k^2(c_1 - c_2k)}{k_1}, \end{aligned}$$

and  $c_1$  and  $c_2$  are arbitrary constants. Similar the last section, the field  $F$  has the form

$$F = \frac{2}{\sqrt{(r_1 - r_3)(r_2 - r_4)}} \left( (r_3 - r_4)E_\pi(S, \frac{r_1 - r_4}{r_1 - r_3}, m) - r_3E_F(S, m) \right), \quad m = \sqrt{\frac{(r_1 - r_4)(r_2 - r_3)}{(r_1 - r_3)(r_2 - r_4)}}, \quad (40)$$

where  $S$  is the Jacobi elliptic function  $S = \text{sn}(\sqrt{(r_1 - r_3)(r_2 - r_4)}X, m)$ ,  $E_F$  and  $E_\pi$  are the first and third incomplete elliptic integrals and  $r_1, r_2, r_3, r_4$  are related with  $a_1, a_2, a_3, a_4$  in the following relations

$$\begin{aligned} a_1 &= -4(r_1 + r_2 + r_3 + r_4), & a_2 &= 4(r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4), \\ a_3 &= -4(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4), & a_4 &= 4r_1r_2r_3r_4. \end{aligned}$$



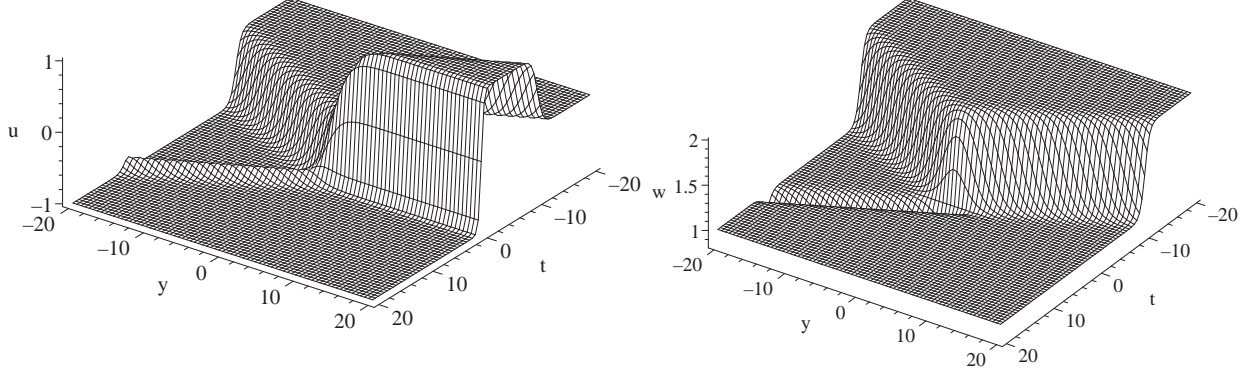


FIG. 2: The interaction solution between a soliton and one-resonant soliton solution for fields  $u$  and  $w$  at  $x = 0$  respectively. The parameters are  $n = 1, k = -1, k_1 = 1, l = 0, \omega = 2$ .

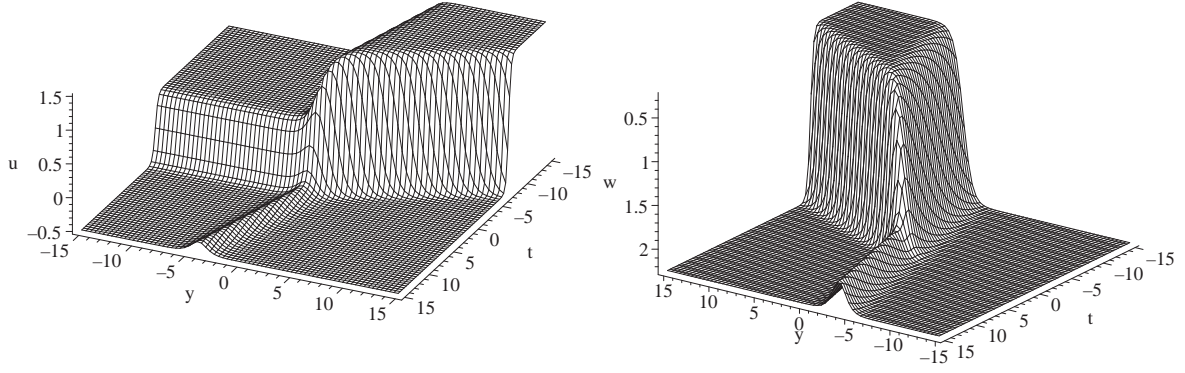


FIG. 3: The interaction solution between a soliton and one-resonant soliton solution for fields  $u$  and  $w$  at  $x = 0$  respectively. The parameters are  $n = 1, k = 1, k_1 = -1, l = 1, \omega = 2$ .

The soliton-multiple wave interaction solution for the mKP equation has the following form

$$f = kx + ly + \omega t + \frac{1}{2} \ln \left( 1 + \sum_{i=1}^n \exp(k_i x + l_i y + \omega_i t) \right), \quad (41)$$

where  $k_i$  are arbitrary constants while  $l_i$  and  $\omega_i$  are determined by the relations

$$l_i = \frac{k_i}{2k} (4k^3 + 6k^2 k_i + 2k k_i^2 \pm 6kl \pm 3k_i l + 2\omega), \quad \omega_i = \pm \frac{k_i}{k} (2k^2 + k k_i \pm l). \quad (42)$$

We can also find other relations to satisfy (33) and do not list it here. In the following, we select the symbol “ $\pm$ ” in (42) as “ $+$ ” to plot the figures. Figures 2 and 3 display the special interaction behavior between a soliton and one-resonant soliton solution with the parameters selected as  $n = 1, k = -1, k_1 = 1, l = 0, \omega = 2$  and  $n = 1, k = 1, k_1 = -1, l = 1, \omega = 2$  respectively. It demonstrate that the interaction behavior is different with selecting different parameters. Figure 4 displays the special interaction behavior between a soliton and two-resonant soliton solutions with the parameters selected as  $n = 2, k = 1, k_1 = 1, k_2 = -2, l = 1, \omega = -1$ .

## V. CONCLUSIONS

In summary, the nonlocal symmetries of the mKP equation are obtained with the truncated Painlevé method. To solve the initial value problem related by the nonlocal symmetries, we prolong

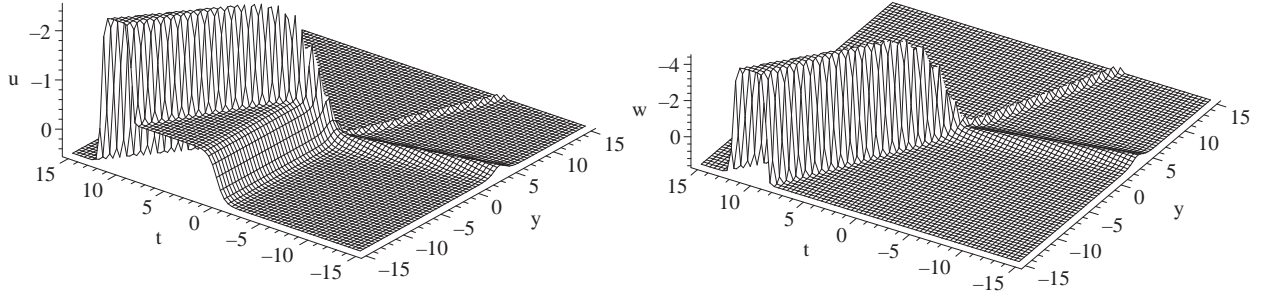


FIG. 4: The interaction solution between a soliton and two-resonant soliton solutions for fields  $u$  and  $w$  at  $x = 0$  respectively. The parameters are  $n = 2, k = 1, k_1 = 1, k_2 = -2, l = 1, \omega = -1$ .

the mKP equation such that nonlocal symmetries becomes the local Lie point symmetries for the prolonged system. The finite symmetry transformations of the prolonged mKP system is derived by using the Lie's first principle. The multi-solitary wave solution for mKP equation is given with the finite symmetry transformations. Thanks to the localization process, the nonlocal symmetries are used to find possible symmetry reductions. The interaction solutions among one soliton and cnoidal waves are given as shown in (24) with (28). In the meanwhile, the CTE method is applied to the mKP equation. With the help of the CTE method, we have found abundant interaction solutions among a soliton and other types of nonlinear waves such as cnoidal periodic waves and multiple resonant soliton solutions. There exist other methods to construct the nonlocal symmetries such as those obtained from the bilinear forms and negative hierarchies [26, 27], nonlinearizations [28], point symmetries [29] and self-consistent sources [30] etc. Using these various nonlocal symmetries to derive new interaction solutions of the nonlinear integrable systems are worthy of further study.

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